Announcements

1) Midterm will be two weeks from today

2) 9-26 notes are now complete

3) Notes from last class - added the proof that the last example is an inner product

Norms from Inner Products

If $\langle \cdot , \cdot \rangle$ is an inner product on V (over $\mathbb{R}$ or $\mathbb{C}$), we can define a norm $\| \cdot \|_2$ on V by

$$\boxed{\| x \|_2 = \langle x, x \rangle^{1/2}}$$

$\forall \; x \in V.$

The first two conditions of a norm are immediate.

The triangle inequality follows from direct calculation.

How to tell when a norm comes from an inner product?

Observe that if $\langle \cdot , \cdot \rangle$ is an inner product on V, then if $x, y \in V$,

$$\|x-y\|_2^2 - \|x+y\|_2^2$$

$$= \langle x-y, x-y \rangle - \langle x+y, x+y \rangle$$

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$$= \langle x, x-y\rangle - \langle y, x-y\rangle - \langle x, x+y\rangle - \langle y, x+y\rangle$$

linearity in 1st coordinate

$$= \cancel{\overbrace{\langle x, x\rangle}^{\|x\|_2^2}} - \langle x, y\rangle - \langle y, x\rangle + \cancel{\langle y, y\rangle} - \cancel{\langle x, x\rangle} - \langle x, y\rangle - \langle y, x\rangle - \cancel{\underbrace{\langle y, y\rangle}_{\|y\|_2^2}}$$

linearity in 2nd coordinate

If V is a real vector space, then

$\langle x, y\rangle = \langle y, x\rangle$ and

we get

$$\|x-y\|_2^2 - \|x+y\|_2^2 = -4\langle x, y\rangle,$$

so

$$\boxed{\langle x, y\rangle = \frac{\|x+y\|_2^2 - \|x-y\|_2^2}{4}}$$

Theorem: (parallelogram property)

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ and let $\|\cdot\|$ be a norm on V. Then $\|\cdot\|$ arises as a norm from an inner product if and only if $\|\cdot\|$ satisfies the parallelogram property: $\forall x, y \in V$,

$$\boxed{\frac{\|x+y\|^2 + \|x-y\|^2}{2} = \|x\|^2 + \|y\|^2}$$

Proof $\Rightarrow$) Suppose $\|\cdot\| = \|\cdot\|_2$ for some inner product $\langle \cdot, \cdot \rangle$ on V.

Then $\forall\, x, y \in V$,

$$\frac{\|x+y\|_2^2 + \|x-y\|_2^2}{2}$$

$$= \frac{\langle x,x\rangle + \cancel{\langle x,y\rangle} + \cancel{\langle y,x\rangle} + \langle y,y\rangle}{2}$$

$$+ \frac{\langle x,x\rangle - \cancel{\langle x,y\rangle} - \cancel{\langle y,x\rangle} + \langle y,y\rangle}{2}$$

$$= \|x\|_2^2 + \|y\|_2^2$$

$\Leftarrow$ Suppose $\|\cdot\|$ on V and that $\|\cdot\|$ satisfies the parallelogram property.

Define for $x, y \in V$

$$\langle x, y \rangle = \frac{\|x+y\|^2 - \|x-y\|^2}{4} \quad (V \text{ real})$$

- or -

$$\langle x, y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2\right)$$

Check inner product properties! $\square$

Example 1 : (p-norms)

If $\|\cdot\|_p$ is the p-norm on $\mathbb{R}^n$ (or $\mathbb{C}^n$) for $1 \leq p \leq \infty$, then $\|\cdot\|_p$ arises from an inner product if and only if $p = 2$.

Idea: Show the parallelogram property fails for $p \neq 2$.

For example, if $n=2$ and $p=\infty$, let $x=e_1$ and $y=e_2$.

Then $\|x\|_\infty = \|y\|_\infty = 1$

Moreover, $\|x-y\|_\infty$

$$= \|(1,-1)\|_\infty$$

$$= 1$$

and $\|x+y\|_\infty = \|(1,1)\|_\infty$

$$= 1 .$$

Then

$$2 = \|x\|_\infty + \|y\|_\infty$$

$$= \|x-y\|_\infty + \|x+y\|_\infty$$

$\neq$ parallelogram property.

This easily generalizes to any n by again using $\{e_1, e_2\}$.

Angles

Given an inner product $\langle \cdot , \cdot \rangle$ on a vector space V (over $\mathbb{R}$ or $\mathbb{C}$) and $x, y \in V$, $x \neq 0_V \neq y$. We can then use the inner product to define the angle Θ between x any y by

$$\boxed{\Theta = \arccos\left(\frac{|\langle x, y \rangle|}{\|x\|_2 \|y\|_2} \right)}$$

Observe that if $x = \alpha y$,

$$\langle x, y \rangle = \langle \alpha y, y \rangle$$
$$= \alpha \|y\|_2^2$$

and $\|x\|_2 = \langle \alpha y, \alpha y \rangle^{1/2}$

$$= \left(|\alpha|^2 \langle y, y \rangle \right)^{1/2}$$

$$= |\alpha| \, \|y\|_2 .$$

Therefore, $\dfrac{|\langle x, y \rangle|}{\|x\|_2 \|y\|_2}$

$$= \frac{|\alpha \|y\|_2^2|}{|\alpha| \cdot \|y\|_2^2}$$

$$= \cancel{\frac{|\alpha| \|y\|_2^2}{|\alpha| \|y\|_2^2}}$$

$$= 1 \quad . \quad \operatorname{Arccos}(1) = 0,$$

So this reflects the fact that x and y are scalar moltiples.

If $x \neq \alpha y$ for any α,

then x and y span a plane (2 dimensional linear space) in V.

The angle then represents the angle between x and y on this plane = triangle geometry.